Modular invariants of compact quantum groups

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Part I: introduction and general situation

Definition (Compact quantum group) [Woronowicz '98]

Compact quantum group $\mathbb{G}$ consists of:

- a unital C*-algebra $\mathfrak{A}$,
- a unital $\star$-homomorphism $\Delta\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ (*comultiplication*) such that:

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta,$$
$$\overline{\mathrm{span}}\, \Delta(\mathfrak{A})(\mathbb{1} \otimes \mathfrak{A}) = \overline{\mathrm{span}}\, \Delta(\mathfrak{A})(\mathfrak{A} \otimes \mathbb{1}) = \mathfrak{A} \otimes \mathfrak{A}.$$

- [Woronowicz, Van Daele] There exisits a unique Haar integral: state $h \in \mathfrak{A}^*$ which is invariant:

$$(h \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes h)\Delta(x) = h(x)\mathbb{1} \quad (x \in \mathfrak{A}).$$

Throughout the talk, I'll assume that h faithful.

- Let $(\mathrm{L}^2(\mathbb{G}), \pi_h, \Omega_h)$ be the GNS representation for h.
- We write $\mathrm{C}(\mathbb{G}) = \pi_h(\mathfrak{A})$, $\mathrm{L}^\infty(\mathbb{G}) = \pi_h(\mathfrak{A})''$, $\mathrm{L}^1(\mathbb{G}) = \mathrm{L}^\infty(\mathbb{G})_*$.

Examples

- Let G be a compact Hausdorff group with Haar measure μ. Define $\mathrm{C}(\mathbb{G}) = \mathrm{C}(G)$ and Δ via

$$\Delta(f)(x,y) = f(xy) \quad (f \in \mathrm{C}(G), x, y \in G).$$

  Then $\mathrm{L}^\infty(\mathbb{G}) = \mathrm{L}^\infty(G), h = \int_G \cdot \, \mathrm{d}\mu$.

- Let Γ be a discrete group. Define $\mathrm{C}(\mathbb{G}) = \mathrm{C}^*_r(\Gamma)$ and

$$\Delta \colon \mathrm{C}^*_r(\Gamma) \ni \lambda_\gamma \mapsto \lambda_\gamma \otimes \lambda_\gamma \in \mathrm{C}^*_r(\Gamma) \otimes \mathrm{C}^*_r(\Gamma).$$

  Then $h(\lambda_\gamma) = \delta_{e,\gamma}$ and $\mathrm{L}^\infty(\mathbb{G}) = \mathrm{L}(\Gamma)$. We write $\mathbb{G} = \widehat{\Gamma}$.

- With any quantum group $\mathbb{G}$ we can associate its dual $\widehat{\mathbb{G}}$ and $\widehat{\widehat{\mathbb{G}}} \simeq \mathbb{G}$. If $\mathbb{G}$ is compact, $\widehat{\mathbb{G}}$ is discrete.

Example: $\mathbb{G} = \mathrm{SU}_q(2)$ $(0 < q < 1)$

- $\mathrm{C}(\mathrm{SU}_q(2))$ is defined as the universal unital C^*-algebra generated by $\alpha, \gamma \in \mathrm{C}(\mathrm{SU}_q(2))$ such that

$$\begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix} \quad \text{is unitary.}$$

- $\Delta \colon \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{C}(\mathrm{SU}_q(2)) \otimes \mathrm{C}(\mathrm{SU}_q(2))$ acts via

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma,$$
$$\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

- $\mathrm{L}^\infty(\mathrm{SU}_q(2)) = \pi_h(\mathrm{C}(\mathrm{SU}_q(2)))''$.

Modular theory of $\mathbb{G}$

- There is a point-w^* continuous group of modular automorphisms $\sigma_t^h \in \mathrm{Aut}(\mathrm{L}^\infty(\mathbb{G}))\,(t \in \mathbb{R})$

$$h(x\,y) = h(y\,\sigma^h_{-i}(x)) \qquad (x, y \in \mathrm{L}^\infty(\mathbb{G}) \text{ nice}).$$

- There is a point-w^* continuous group of scaling automorphisms $\tau_t \in \mathrm{Aut}(\mathrm{L}^\infty(\mathbb{G}))$. $\tau_{-i} = S_{\mathbb{G}}^2$, where $S_{\mathbb{G}}$ is the antipode.

Modular theory of $\mathbb{G}$

- Compact quantum groups have representation theory resembling the classical one.
- $\mathrm{Irr}(\mathbb{G})$ – set of (classes of) irreducible representations

$$\alpha \in \mathrm{Irr}(\mathbb{G}) \quad \rightsquigarrow \quad U^{\alpha} \in \mathrm{C}(\mathbb{G}) \otimes \mathrm{B}(\mathsf{H}_{\alpha}), \quad \dim \mathsf{H}_{\alpha} < +\infty.$$

$$\{\xi_i^{\alpha}\}_{i=1}^{\dim(\alpha)} \text{ orthonormal basis of } \mathsf{H}_{\alpha} \rightsquigarrow U_{i,j}^{\alpha} = (\mathrm{id} \otimes \omega_{\xi_i^{\alpha}, \xi_j^{\alpha}}) U^{\alpha} \in \mathrm{C}(\mathbb{G}).$$

- There is a family of positive, invertible operators $\rho_{\alpha} \in \mathrm{B}(\mathsf{H}_{\alpha})$.
- Automorphisms σ_t^h, τ_t can be expressed using ρ_{α}.

$$\sigma_t^h(U_{i,j}^{\alpha}) = (\rho_{\alpha,i}\rho_{\alpha,j})^{it} U_{i,j}^{\alpha}, \quad \tau_t(U_{i,j}^{\alpha}) = \left(\frac{\rho_{\alpha,i}}{\rho_{\alpha,j}}\right)^{it} U_{i,j}^{\alpha},$$

  where $\rho_{\alpha} = \mathrm{diag}(\rho_{\alpha,1}, \ldots, \rho_{\alpha,\dim(\alpha)})$.
- h is tracial $\Leftrightarrow \forall_t \, \sigma_t^h = \mathrm{id} \Leftrightarrow \forall_{\alpha} \, \rho_{\alpha} = \mathbb{1} \Leftrightarrow \forall_t \, \tau_t = \mathrm{id}$.
  In this case $\mathbb{G}$ is of Kac type.

Examples

- If $\mathbb{G} = G$ then $h = \int_G \cdot \, \mathrm{d}\mu$ hence $\sigma^h_t = \tau_t = \mathrm{id}$.
- If $\mathbb{G} = \widehat{\Gamma}$ with Γ discrete (so $\mathrm{L}^\infty(\mathbb{G}) = \mathrm{L}(\Gamma)$) then $h(\lambda_\gamma) = \delta_{\gamma,e}$ hence also $\sigma^h_t = \tau_t = \mathrm{id}$.
- If $\mathbb{G} = \mathrm{SU}_q(2)$ then the Haar integral h is not tracial and

$$\sigma^h_t(\alpha) = q^{-2it}\alpha, \quad \sigma^h_t(\gamma) = \gamma,$$
$$\tau_t(\alpha) = \alpha, \quad \tau_t(\gamma) = q^{2it}\gamma.$$

Modular invariants – motivation

- Fix $0 < \lambda < 1$. With Piotr we've constructed a family of CQGs $\{\mathbb{K}_j\}_{j\in\mathbb{J}}$ such that $\mathrm{L}^\infty(\mathbb{K}_j)$ is the injective type III_λ factor.
- How can we show that that $\mathbb{K}_j \not\simeq \mathbb{K}_{j'}$?
- $\{1,\alpha_j\}_{j\in\mathbb{J}}$ – basis of $\mathbb{R}$ over $\mathbb{Q}$, $\Gamma_j = \alpha_j \frac{2\pi}{\log(\lambda)}\mathbb{Z}$,

$$\mathbb{K}_j = \Gamma_j \bowtie \mathbb{G} \;\text{ via } \Gamma_j \times \mathrm{L}^\infty(\mathbb{G}) \ni (\gamma, x) \mapsto \tau_\gamma(x) \in \mathrm{L}^\infty(\mathbb{G}).$$

- $\tau_t \in \mathrm{Inn}(\mathrm{L}^\infty(\mathbb{K}_j))$ if and only if $t \in \Gamma_j + \frac{2\pi}{\log(\lambda)}\mathbb{Z}$

$$\Rightarrow \; \mathbb{K}_j \not\simeq \mathbb{K}_{j'} \; (j \neq j').$$

Let $\mathbb{G}$ be a compact quantum group.

[K., Sołtan '23] Modular invariants

Define subgroups of $\mathbb{R}$:

$$
\begin{aligned}
T^{\tau}(\mathbb{G}) &= \{t \in \mathbb{R} \,|\, \tau_t = \mathrm{id}\}, \\
T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) &= \{t \in \mathbb{R} \,|\, \tau_t \in \mathrm{Inn}(\mathrm{L}^{\infty}(\mathbb{G}))\}, \\
T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) &= \{t \in \mathbb{R} \,|\, \tau_t \in \overline{\mathrm{Inn}}(\mathrm{L}^{\infty}(\mathbb{G}))\}.
\end{aligned}
$$

And similarly $T^{\sigma}(\mathbb{G}) = \{t \in \mathbb{R} \,|\, \sigma^h_t = \mathrm{id}\}$, $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}), T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G})$.

- $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T(\mathrm{L}^{\infty}(\mathbb{G}))$ is the Connes' T-invariant.

Let $\mathbb{G}$ be a locally compact quantum group.

[K., Sołtan '23] Modular invariants

Define subgroups of $\mathbb{R}$:

$$T^{\tau}(\mathbb{G}) = \{t \in \mathbb{R} \,|\, \tau_t = \mathrm{id}\},$$
$$T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \{t \in \mathbb{R} \,|\, \tau_t \in \mathrm{Inn}(\mathrm{L}^{\infty}(\mathbb{G}))\},$$
$$T^{\tau}_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \{t \in \mathbb{R} \,|\, \tau_t \in \overline{\mathrm{Inn}}(\mathrm{L}^{\infty}(\mathbb{G}))\}.$$

And similarly $T^{\sigma}(\mathbb{G}) = \{t \in \mathbb{R} \,|\, \sigma^{\varphi}_t = \mathrm{id}\}$, $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}), T^{\sigma}_{\overline{\mathrm{Inn}}}(\mathbb{G})$,

$$\mathrm{Mod}(\mathbb{G}) = \{t \in \mathbb{R} \,|\, \delta^{it} = \mathbb{1}\} \text{ where } \delta \eta \mathrm{L}^{\infty}(\mathbb{G}) \text{ is the modular element.}$$

- $T^{\sigma}_{\mathrm{Inn}}(\mathbb{G}) = T(\mathrm{L}^{\infty}(\mathbb{G}))$ is the Connes' T-invariant.
- All these sets depends only on the isomorphism class of $\mathbb{G}$.

Let $\mathbb{G}$ be a locally compact quantum group.

Modular invariants

- A priori we obtain 14 subgroups of $\mathbb{R}$:

$$T^{\tau}, T^{\tau}_{\mathrm{Inn}}, T^{\tau}_{\overline{\mathrm{Inn}}}, T^{\sigma}, T^{\sigma}_{\mathrm{Inn}}, T^{\sigma}_{\overline{\mathrm{Inn}}} \text{ and } \mathrm{Mod} \text{ for } \mathbb{G}, \widehat{\mathbb{G}}.$$

- There are easy reductions:

$$T^{\tau}(\mathbb{G}) = T^{\tau}(\widehat{\mathbb{G}}), \quad T^{\sigma}(\mathbb{G}) = T^{\tau}(\mathbb{G}) \cap \mathrm{Mod}(\widehat{\mathbb{G}})$$

  so we have 11 subgroups. It follows from

  - $(\tau_t \otimes \widehat{\tau}_t)\mathrm{W}^{\mathbb{G}} = \mathrm{W}^{\mathbb{G}}$,
  - $\nabla_{\psi}^{it} = \widehat{\delta}^{-it} P^{-it}$ (P^{it} implements τ_t and $\widehat{\tau}_t$).

Modular invariants

- If $\mathbb{G}$ is compact then $\delta = \mathbb{1}, \ell^\infty(\widehat{\mathbb{G}}) = \prod_{\alpha \in \mathrm{Irr}(\mathbb{G})} \mathrm{B}(\mathsf{H}_\alpha)$ so we are left with 6 invariants

$$T^\tau(\mathbb{G}), T^\tau_{\mathrm{Inn}}(\mathbb{G}), T^\tau_{\overline{\mathrm{Inn}}}(\mathbb{G}), T^\sigma_{\mathrm{Inn}}(\mathbb{G}), T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}), \mathrm{Mod}(\widehat{\mathbb{G}}).$$

- If additionally $\mathrm{L}^\infty(\mathbb{G})$ is semifinite, then $T^\sigma_{\mathrm{Inn}}(\mathbb{G}) = T^\sigma_{\overline{\mathrm{Inn}}}(\mathbb{G}) = \mathbb{R}$ and there are 4 possibly non-trivial invariants. This is the case for G_q.

$\mathbb{G}$ is second countable $\Leftrightarrow$ $\mathrm{C}(\mathbb{G})$ is separable.

[K., Sołtan '23] Conjecture

Let $\mathbb{G}$ be a second countable compact quantum group. Assume $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$. Is $\mathbb{G}$ of Kac type?

- Equivalently: $\mathbb{G}$ second countable, not of Kac type. Do we have $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \neq \mathbb{R}$?
- [K., Sołtan '23] The answer is affirmative in special cases:
  - there is a unitary representation U with $2 = \dim(U) < \dim_q(U)$,
  - $\mathrm{C}^u(\mathbb{G})$ is type I, in particular $\mathbb{G} = G_q$,
  - $\mathbb{G} = U_F^+$,
  - $\widehat{\mathbb{G}}$ satisfies an ICC-type condition.

Part II: q-deformations

Lie group G and companions

- Let G be a simply connected, semisimple, compact Lie group with:
  - complexified Lie algebra $\mathfrak{g}$,
  - maximal torus $\mathbb{T}^r \simeq T \subseteq G$ with complexified Lie algebra $\mathfrak{h}$
  - root system $\Phi \subseteq \mathfrak{h}^*$, positive roots Φ^+, simple roots $\alpha_1, \ldots, \alpha_r \in \Phi^+$,
  - Weyl group W and W-invariant $\langle \cdot | \cdot \rangle$ form on $\mathfrak{h}$ such that $\langle \alpha | \alpha \rangle = 2$ for short roots α.
  - Weyl vector $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in \mathbf{P}^+$,
  - root and weight lattice $\mathbf{Q} \subseteq \mathbf{P} \subseteq \mathfrak{h}^*$,
  - positive cone $\mathbf{P}^+ \subseteq \mathbf{P}$.

For $SU(n+1)$: $r = n, W = S_{n+1}$,

$$\mathbf{P} = \{(\lambda_1, \ldots, \lambda_{n+1}) \,|\, \lambda_i - \lambda_j \in \mathbb{Z}\} / \mathbb{R}(1, \ldots, 1) \simeq \mathbb{Z}^n,$$

$$\mathbf{Q} = \{(\lambda_1, \ldots, \lambda_{n+1}) \,|\, \lambda_i \in \mathbb{Z}, \sum_{i=1}^{n+1} \lambda_i = 0\} / \mathbb{R}(1, \ldots, 1), \; \mathbf{P}/\mathbf{Q} \simeq \mathbb{Z}_{n+1},$$

$$\mathbf{P}^+ = \{(\lambda_1, \ldots, \lambda_{n+1}) \,|\, \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+\} / \mathbb{R}(1, \ldots, 1).$$

Fix $0 < q < 1$.

q-DEFORMED ENVELOPING ALGEBRA OF $\mathfrak{g}$

- $U_q\mathfrak{g}$ is the unital algebra generated by $E_i, F_i, K_i, K_i^{-1}\,(1 \leqslant i \leqslant r)$ satisfying certain relations.
- $U_q\mathfrak{g}$ has structure of a Hopf $\star$-algebra. If $\pi\colon U_q\mathfrak{g} \to \mathrm{B}(\mathscr{H})$ is a $\star$-representation, then $\xi \in \mathscr{H}$ has weight $\mathrm{wt}(\xi) \in \mathbf{P}$ if

$$\forall_{1 \leqslant i \leqslant r}\, \pi(K_i)\xi = q^{\langle \mathrm{wt}(\xi) | \alpha_i \rangle}\xi.$$

- $\mathbf{P}^+ \leftrightarrow$ (Finite-dimensional irreducible representations which are direct sums of weight spaces): $\varpi \mapsto (\mathscr{H}_\varpi, \pi_\varpi)$
- $\mathrm{Pol}(G_q) = \mathrm{span}\{$matrix coefficients of π_ϖ as above$\} \subseteq (U_q\mathfrak{g})^*$.

Compact quantum group G_q

- We complete $\mathrm{Pol}(G_q)$ to a C*-algebra $\mathrm{C}(G_q)$ and G_q is a CQG.
- $\mathrm{Irr}(G_q) = \mathbf{P}^+$, $\mathrm{Pol}(G_q)$ is spanned by

$$U^{\varpi}(\xi,\eta) \quad (\varpi \in \mathbf{P}^+, \xi, \eta \in \mathscr{H}_{\varpi},\ \mathrm{wt}(\xi), \mathrm{wt}(\eta) \in \mathbf{P}).$$

- Pairing $U_q\mathfrak{g} \times \mathrm{Pol}(G_q) \to \mathbb{C}$ is given by

$$\langle x, U^{\varpi}(\xi,\eta)\rangle = \langle \xi | \pi_{\varpi}(x)\eta\rangle.$$

AUTOMORPHISMS FOR G_q

- $\sigma^h_t(U^{\varpi}(\xi,\eta)) = q^{\langle 2\rho|\,\mathrm{wt}(\xi)+\mathrm{wt}(\eta)\rangle it}\, U^{\varpi}(\xi,\eta)$,
- $\tau_t(U^{\varpi}(\xi,\eta)) = q^{\langle 2\rho|\,\mathrm{wt}(\xi)-\mathrm{wt}(\eta)\rangle it}\, U^{\varpi}(\xi,\eta)$.

where $\rho \in \mathbf{P}^+$ is the Weyl vector and $\langle\cdot|\cdot\rangle$ is the W-invariant form on $\mathfrak{h}^*$.

C^*-ALGEBRA $\mathrm{C}(G_q)$

- [Soibelman] Irreducible representations of $\mathrm{C}(G_q)$ are (up to equivalence) precisely

  $$\pi_{\lambda,w} = \pi_\lambda \star \pi_w = (\pi_\lambda \otimes \pi_w)\Delta\colon \mathrm{C}(G_q) \to \mathrm{B}(\ell^2(\mathbb{Z}_+)^{\otimes \ell(w)}) \quad ((\lambda, w) \in T \times W)$$

  where

  $$\pi_\lambda\colon \mathrm{C}(G_q) \ni U^{\varpi}(\xi,\eta) \mapsto \langle \xi|\eta\rangle \langle \mathrm{wt}(\xi), \lambda\rangle \in \mathbb{C}$$

  are characters and

  $$\pi_w\colon \mathrm{C}(G_q) \to \mathrm{B}(\ell^2(\mathbb{Z}_+)^{\otimes \ell(w)})$$

  are built using $\mathrm{C}(G_q) \to \mathrm{C}(\mathrm{SU}_{q_i}(2)) \to \mathrm{B}(\ell^2(\mathbb{Z}_+))$.
- $\mathrm{C}(G_q)$ is type I.

Haar integral on G_q

- [Reshetikhin-Yakimov '01] Haar integral on G_q can be calculated as

$$h(x) = \Big(\prod_{\alpha\in\Phi^+}(1-q^{2\langle\rho|\alpha\rangle})\Big)\int_T \operatorname{Tr}\big(\pi_{\lambda,w_\circ}(x|b_\rho|^2)\big)\,\mathrm{d}\lambda \quad (x\in \mathrm{C}(G_q))$$

  where $\mathrm{d}\lambda$ is normalised Lebesgue measure on T, $w_\circ$ is the longest element in W,
- $b_\rho = U^\rho(\xi_\rho,\eta_{w_\circ\rho})\in \operatorname{Pol}(G_q)$ (equal to $-\gamma$ for $\mathrm{SU}_q(2)$).
- Desmedt's theorem: we obtain:
  - unitary $\mathcal{Q}_L\colon \mathrm{L}^2(G_q)\to\int_T^\oplus \mathrm{HS}(\mathscr{H}_\lambda)\,\mathrm{d}\lambda$ such that
  - $\mathcal{Q}_L\,\mathrm{L}^\infty(G_q)\mathcal{Q}_L^* = \int_T^\oplus \mathrm{B}(\mathscr{H}_\lambda)\otimes \mathbb{1}_{\overline{\mathscr{H}_\lambda}}\,\mathrm{d}\lambda$.

[K., Sołtan '23] Scaling group

For $t\in\mathbb{R}, x=\int_T^\oplus x_\lambda\otimes \mathbb{1}_{\overline{\mathscr{H}_\lambda}}\mathrm{d}\lambda\in \mathrm{L}^\infty(G_q)$ we have

$$\tau_t(x)=\int_T^\oplus \pi_{w_\circ}(|b_\rho|^{-2it})x_{\lambda\boldsymbol{\lambda}_{2t}}\pi_{w_\circ}(|b_\rho|^{2it})\otimes\mathbb{1}_{\overline{\mathscr{H}_\lambda}}\mathrm{d}\lambda$$

where $\mathbb{R}\ni t\mapsto\boldsymbol{\lambda}_t\in T$ is given by $\langle\varpi,\boldsymbol{\lambda}_t\rangle=q^{\langle 2\rho|\varpi\rangle it}\,(\varpi\in\mathbf{P}\simeq\widehat{T})$.

- Corollary: $T^\tau_{\mathrm{Inn}}(G_q)=T^\tau_{\overline{\mathrm{Inn}}}(G_q)$.

Lemma

- $\{\langle 2\rho|\alpha\rangle \,|\, \alpha \in \mathbf{Q}\} = 2\mathbb{Z}$,
- $\{\langle 2\rho|\varpi\rangle \,|\, \varpi \in \mathbf{P}\} = \Upsilon_\Phi\mathbb{Z}$ is a nontrivial subgroup of $\mathbb{Z}$ ($\Upsilon_\Phi \in \mathbb{N}$).

[K., Sołtan '23] Theorem

Modular invariants for G_q are given by

$$T^\tau(G_q) = \tfrac{\pi}{\log(q)}\mathbb{Z}, \quad T^\tau_{\mathrm{Inn}}(G_q) = T^{\tau}_{\overline{\mathrm{Inn}}}(G_q) = \mathrm{Mod}(\widehat{G_q}) = \tfrac{\pi}{\Upsilon_\Phi \log(q)}\mathbb{Z}.$$

Consequently Conjecture holds for G_q.

Irreducible root systems

- Every root system Φ decomposes as $\Phi_1 \oplus \cdots \oplus \Phi_l$ for Φ_i irreducible.
- $\Upsilon_\Phi = \gcd(\Upsilon_{\Phi_1}, \ldots, \Upsilon_{\Phi_l})$.
- Irreducible root systems are classifed:
  - type A_n $(n \geqslant 1), G = \mathrm{SU}(n+1)$,
  - type B_n $(n \geqslant 2), G = \mathrm{Spin}(2n+1)$,
  - type C_n $(n \geqslant 3), G = \mathrm{Sp}(2n)$,
  - type D_n $(n \geqslant 4), G = \mathrm{Spin}(2n)$,
  - exceptional types: E_6, E_7, E_8, F_4, G_2.

$T^\tau(G_q) = \frac{\pi}{\log(q)}\mathbb{Z}$, $T^\tau_{\mathrm{Inn}}(G_q) = \frac{\pi}{\Upsilon_\Phi \log(q)}\mathbb{Z}$.

[K., Sołtan '23] Invariants in simple case

- $A_n (n \geqslant 1)$, $G_q = \mathrm{SU}_q(n+1)$: $\Upsilon_\Phi = 1\,(n \text{ odd}), \Upsilon_\Phi = 2\,(n \text{ even})$.
  - In particular $\Upsilon_\Phi = 2$ for $\mathrm{SU}_q(3)$.
- $B_n (n \geqslant 2)$: $\Upsilon_\Phi = 1\,(n \text{ odd}), \Upsilon_\Phi = 2\,(n \text{ even})$.
- $C_n (n \geqslant 3)$: $\Upsilon_\Phi = 2$.
- $D_n (n \geqslant 4)$: $\Upsilon_\Phi = 2\,(n \in 4\mathbb{N} + \{0,1\}), \Upsilon_\Phi = 1\,(n \in 4\mathbb{N} + \{2,3\})$.
- For E_6, E_7, E_8, F_4, G_2 number Υ_Φ is equal to $2, 1, 2, 2, 2$.

$\dim(\varpi) \geqslant 2$ for non-trivial $\varpi \in \mathbf{P}^+$.

Corollary

If $\Upsilon_\Phi \geqslant 2$, then G_q has non-trivial, inner scaling automorphisms not implemented by a group-like unitary.

[K., Sołtan '23] Inner scaling automorphism

- Choose G such that $\Upsilon_\Phi = 2$ and set $\boldsymbol{t}_0 = \frac{\pi}{2\log(q)}$. Then $\tau_{\boldsymbol{t}_0} \in \operatorname{Aut}(G_q)$ is inner, non-trivial, not implemented by a group-like unitary.
- $\tau_{\boldsymbol{t}_0}$ is implemented by unitary $\int_T^{\oplus} \pi_{w_\circ}(|b_\rho|)^{-2i\boldsymbol{t}_0} \otimes \mathbb{1}_{\overline{\mathscr{H}_\lambda}} \,\mathrm{d}\lambda$.
- If $\tau_{\boldsymbol{t}_0} = \operatorname{Ad}(v)$ then $v \notin \mathrm{C}(G_q)$. Consequently $\tau_{\boldsymbol{t}_0}\restriction_{\mathrm{C}(G_q)}$ is not inner.

These results hold in particular for $G_q = \mathrm{SU}_q(3)$.

Part III: Special case of conjecture

[K., Sołtan '23] Theorem

Let $\mathbb{G}$ be a second countable compact quantum group, assume:

- there is a finite dimensional unitary representation U with $2 = \dim(U) < \dim_q(U)$.

Then $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) \neq \mathbb{R}$.

- Assume by contradiction $T^{\tau}_{\mathrm{Inn}}(\mathbb{G}) = \mathbb{R}$. Then $\tau_t = \mathrm{Ad}(a^{it})$ for strictly positive $a \,\eta\, \mathrm{L}^{\infty}(\mathbb{G})$.
- There is a family of irreducible representations $(U^n)_{n\in\mathbb{N}}$ such that:
  - $U^1 = U$,
  - $$\gamma(U^n) = \gamma(U)^{2n}, \Gamma(U^n) = \Gamma(U)^{2n}, \gamma(U^n) = \Gamma(U^n)^{-1}$$
    where $\gamma(U^n), \Gamma(U^n)$ is the smallest/largest eigenvalue of ρ_{U^n}.
  - $\inf_{n\in\mathbb{N}} \frac{\Gamma(U^n)}{\dim_q(U^n)} > 0$.

- Set $\mathbb{H} = \mathbb{G} \times \mathbb{G}$, write $\|x\|_2 = h_{\mathbb{H}}(x^* x)^{1/2}$.
- Set $\varepsilon_t = \|a^{it} \otimes a^{it} - \mathbb{1} \otimes \mathbb{1}\|_2$ for $t \in \mathbb{R}$ and $X_n = {U^n}_{1,\dim U^n} \otimes \overline{U^n}_{\dim U^n,1}$.
- We have $\sigma_t^{h_{\mathbb{H}}}(X_n) = X_n, \tau_t^{\mathbb{H}}(X_n) = \Gamma(U^n)^{-4it} X_n$ and

$$\|X_n\|_2 = \frac{\Gamma(U^n)}{\dim_q(U^n)} \geqslant \inf_{m \in \mathbb{N}} \frac{\Gamma(U^m)}{\dim_q(U^m)} = c > 0.$$

- Using $\tau_t^{\mathbb{H}} = \mathrm{Ad}(a^{it} \otimes a^{it})$ we obtain

$$\begin{aligned} |\Gamma(U^n)^{-4it} - 1| c &\leqslant |\Gamma(U^n)^{-4it} - 1| \|X_n\|_2 = \|\tau_t^{\mathbb{H}}(X_n) - X_n\|_2 \\ &= \|(a^{it} \otimes a^{it}) X_n (a^{-it} \otimes a^{-it}) - X_n\|_2 \\ &\leqslant \cdots \leqslant 4\varepsilon_t. \end{aligned}$$

$$\varepsilon_t \xrightarrow[t \searrow 0]{} 0,\ \Gamma(U^n) = \Gamma(U)^{2n} \xrightarrow[n \to \infty]{} +\infty \ \rightsquigarrow\ \text{contradiction.}$$